MODEL-COMPLETENESS AND DECIDABILITY OF THE ADDITIVE STRUCTURE OF INTEGERS EXPANDED WITH A FUNCTION FOR A BEATTY SEQUENCE

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Abstract. We introduce a complete axiomatization for the structure $\mathcal{Z}_\alpha = \langle \mathbb{Z}, +, 0, 1, f\rangle$ where $f : x \mapsto \lfloor \alpha x \rfloor$ is a unary function with α a fixed transcendental number. This result fits into the more general theme of adding traces of multiplication to integers without losing decidability. When α is either a quadratic number or an irrational number less than one, similar decidability results have been already obtained by applying heavy techniques from the theory of autamota. Nevertheless, our approach is based on a clear axiomatization and involves only elementary techniques from model theory.

1. Introduction

The subject of this study is the structure $\mathcal{Z}_\alpha = \langle \mathbb{Z}, +, 0, 1, f\rangle$ which contains the integer addition together with a *trace* of multiplication; namely the function $\lfloor \alpha x \rfloor$ whose range is the Beatty sequence with modulo α.

Our results lie in the intersection of two active research programs. On the one hand, it relates to the recent works on decidability of the expansions of $\langle \mathbb{Z}, +\rangle$ as well as their classification either as stable structures, as in [C18] and [CP18], or unstable structures as in [KS17]. In this sense, $\mathcal{Z}_\alpha$ is an instance of an unstable yet decidable expansion of $\langle \mathbb{Z}, +\rangle$.

On the other hand, $\mathcal{Z}_\alpha$ is definable in the structure $\mathcal{R}_\alpha = \langle \mathbb{R}, <, +, 0, \mathbb{Z}, \alpha\mathbb{Z}\rangle$ which lies in the more general theme of research studying the expansions of real line with specific discrete additive subgroups. Most relevant to our work is Hieronymi's theorem in [H16] which shows, for the special case of a quadratic α, that the theory of the structure $\mathcal{R}_\alpha$—and as a result $\mathcal{Z}_\alpha$—is decidable. Decidability is proved there by showing that $\mathcal{R}_\alpha$ is definable in the monadic second-order structure $\langle \mathbb{N}, P(\mathbb{N}), \in, x \mapsto x + 1\rangle$ which was already known to be decidable ([B62]).

The results on $\mathcal{Z}_\alpha$ have been recently generalized in several directions. In particular, it is shown in [H21] that the common theory of the structures $\mathcal{Z}_\alpha$ is decidable when α ranges over irrational numbers less than one. The proof appeared there applies heavy techniques from the

Date: October 18, 2022.
2020 *Mathematics Subject Classification.* Primary 03B25; Secondary 03C10, 11U09, 11U05.
Key words and phrases. Additive Group of Integers, Beatty sequence, Kronecker's lemma, Decidability, Model-completeness.

theory of automata; wherein a main feature is designing a Büchi automaton which can perform addition over Ostrowski numeration systems.

The current study is indeed along similar lines to a result by the first and the third authors in [KZ22] in which they applied only simple techniques of model theory to provide an alternative proof for the decidability of $\mathcal{Z}_\alpha$ in the case that α is the golden ratio.

In the present paper, by applying elementary tools in model theory and number theory, we prove that the theory of $\mathcal{Z}_\alpha$ is decidable when α is a transcendental number. To our knowledge, in the latter case (transcendental α) nothing is known about the decidability of the more general structure $\mathcal{R}_\alpha$, and we believe that our result forms an important step towards solving that problem as well.

We will discuss the definable sets of $\mathcal{Z}_\alpha$ later in Subsection 5.2. But, to have a general picture of the model theory involved in $\mathcal{Z}_\alpha$, note in particular that $\mathcal{Z}_\alpha$ is a model of the theory of $\mathbb{Z}$-groups, or Presburger arithmetic without an order, and hence it defines all congruence classes or arithmetic progressions. The latter sets are a typical example of definable sets in $\mathcal{Z}_\alpha$ with a somewhat "structured" nature.

On the other hand, there are definable sets in $\mathcal{Z}_\alpha$ with a "random" behaviour, and these sets are typically the subsets of $\mathcal{Z}_\alpha$ defined using the powers of the function f. An aspect of this randomness is reflected in the fact that these sets do not contain an infinite arithmetic progression (see Subsection 5.2 for more details). However, this random behaviour is actually a consequence of definability of a liner order which turns out to be dense by means of Kronecker's approximation lemma (Fact 3.1).

When applied to the simple case of an irrational number α, Kronecker's lemma says that the set of decimal parts of the sequence $\{\alpha n : n \in \omega\}$ is dense in the unit interval $(0, 1)$.

However, as it will be expanded later, we need the full strength of Kronecker's lemma to tackle the case of a transcendental number. In fact, when α is transcendental all different powers of f, and hence all different powers of α, take an independent part in creating the random behaviour. This in fact contrasts the case of a quadratic α where this randomness is actually weaker, mainly because all terms of the language reduce to a linear form; for example, $f^2(x)$ is equal to $f(x) + x - 1$ when α is the golden ratio (see [KZ22]).

The following motivating propositions–that won't directly be used in the paper–reflect the basic idea on which most of our arguments rely. To solve a given system of equations in $\mathcal{Z}_\alpha$ we turn, whenever possible, equations into inequalities in terms of the decimal parts. The solution to the system will then be obtained by an application of k-dimensional Kronecker's lemma, Fact 3.1, asserting the density of the set $\{([\beta_1 n], \dots, [\beta_k n])\}_{n\in\mathbb{N}}$ in $(0, 1)^k$ for a $\mathbb{Q}$-linear independent set of real numbers $\beta_1, \dots, \beta_k$. Recall that the decimal part of a real number α is denoted by $[\alpha]$, namely $[\alpha] := \alpha - \lfloor \alpha \rfloor$.

Proposition 1.1. *Suppose that α is an irrational number and let a and b be integers. Then,*

(1) *a is in the range of f if and only if $\left[\frac{a}{\alpha}\right]$ is greater than $1-\frac{1}{\alpha}$. Moreover, $a = f(b)$ implies that $a = \left\lfloor\frac{b}{\alpha}\right\rfloor + 1$. If α is positive and less than 1, then f is a surjection.*

(2) *$f(a+b) = f(a) + f(b) + \ell$ for some $\ell \in \{0,1\}$, where ℓ is equal to 0 if and only if $[\alpha a] + [\alpha b] < 1$.*

Proof. Part (2) is obvious, for part (1) use the fact that $a = f(b)$ implies $\alpha b - 1 < a < \alpha b$. □

Notice that the language $\mathcal{L}$ that we will be working in won't contain the congruence relations $\overset{m}{\equiv}$, yet the following proposition in given only for its motivating aspect.

Proposition 1.2. *The following system of equations has infinitely many solutions in $\mathbb{Z}$,*

$$\begin{cases} x \overset{m}{\equiv} i \\ f(x) \overset{n}{\equiv} j. \end{cases}$$

Proof. It is easy to verify that $f(x) \overset{n}{\equiv} j$ if and only if $[\frac{\alpha}{n}x] \in (\frac{j}{n}, \frac{j+1}{n})$. Hence, to solve the system above, it suffices to find y such that

$$\left[\frac{(my+i)\alpha}{n}\right] \in (\frac{j}{n}, \frac{j+1}{n}).$$

If $\frac{j+1}{n} > [\frac{i}{n}\alpha]$, by Fact 3.1 we choose y such that

$$\left[\frac{m\alpha}{n}y\right] < 1 - [\frac{i}{n}\alpha], \quad \text{and}$$
$$\frac{j}{n} - \left[\frac{i}{n}\alpha\right] < \left[\frac{m\alpha}{n}y\right] < \frac{j+1}{n} - \left[\frac{i}{n}\alpha\right].$$

In the case that $\frac{j+1}{n} < [\frac{i}{n}\alpha]$, again by Fact 3.1 we choose y such that

$$\left[\frac{m\alpha}{n}y\right] > 1 - \left[\frac{i}{n}\alpha\right], \quad \text{and}$$
$$1 + \frac{j}{n} - \left[\frac{i}{n}\alpha\right] < \left[\frac{m\alpha}{n}y\right] < 1 + \frac{j+1}{n} - \left[\frac{i}{n}\alpha\right].$$

□

We will gradually present a theory $\mathcal{T}_\alpha$ for $\mathcal{Z}_\alpha$, where we introduce each axiom scheme after proving the desired property for $\mathcal{Z}_\alpha$. So each of the sections below will contain axioms that ensure a partial model-completeness for the final theory $\mathcal{T}_\alpha$. We finish by our Main Theorem showing that $\mathcal{T}_\alpha$ is model-complete and this suffices for $\mathcal{Z}_\alpha$ to be decidable. We find it helpful to give a summary of our arguments leading to the proof of model-completeness as follows.

Step 1. We treat certain relations among the decimal parts as first-order $\mathcal{L}$-formulas (Section 2).

Step 2. We divide systems of equations in $\mathcal{L}$ into two main categories of non-algebraic (Section 3) and algebraic formulas (Section 4).

Step 3. We use an extended version of Kronecker lemma (Theorem 3.3) to show that solvability of a system of non-algebraic formulas is equivalent to a quantifier-free $\mathcal{L}$-formula (Theorem 3.9).

Step 4. For two models $\mathcal{M}_1 \subseteq \mathcal{M}_2$, we show that the solution, in $\mathcal{M}_2$, of an algebraic system involving only a single variable and with parameters in $\mathcal{M}_1$, belongs also to $\mathcal{M}_1$ (Lemma 4.3). This implicitly shows that a substructure of a model of $\mathcal{T}_\alpha$ is closed under taking inverse of different powers of f.

Step 5. We use a technical trick (Lemma 4.7) to show that an algebraic system which contains more than two variables reduces to a non-algebraic system of smaller number of variables (Subsections 4.2 and 4.3).

Our last section will contain some additional observations and remarks.

Convention.

(1) α is a fixed transcendental number.

(2) We will be working in the language $\mathcal{L} = \{+, -, 0, 1, f\}$ and unless we state otherwise all formulas are assumed to be in $\mathcal{L}$. In particular, all axioms and axiom schemes are $\mathcal{L}$-formulas.

(3) When there is no mention of a model or a theory, the lemmas and theorems below concern the structure $\mathcal{Z}_\alpha$, and hence the variables and parameters will range over the set of integers $\mathbb{Z}$.

(4) We occasionally use a finite partial type $\Gamma(x)$ as a conjunction of $\mathcal{L}$-formulas as well; that is, we freely use notations like $\exists x \Gamma(x)$ instead of writing $\exists x \bigwedge_{\varphi(x) \in \Gamma(x)} \varphi(x)$.

A note on overlapping results. Short before submitting this paper, we learnt of a similar independent work by Günaydin and Özsahakyan uploaded in arXiv [GO22]. The main difference between the two papers, is that in [GO22] the authors consider the Beatty sequence as a predicate in the language, where we put the function $f = \lfloor \alpha x \rfloor$, which is not definable in their structure. For the same reason, we have to deal with decimals that concern the powers of the function f where they need only to treat decimals of the linear combinations.

2. Describing decimals in $\mathcal{L}$

Although our language/theory does not literally contain the decimals themselves, similar observations to Proposition 1.1 show that our theory is capable enough to describe their key properties in a nice way. In fact, it suffices for $\mathcal{L}$ to capture the order and the dense distribution of these decimals in the spirit of Fact 3.1 below. Through the following two lemmas we first show to what extent the properties of the decimals are expressible in $\mathcal{L}$.

Lemma 2.1. *Let a and b be integers, and $n \in \mathbb{N}$ with $n \neq 0$.*

(1) There exists a natural number $i \in \{0, \dots, n-1\}$ such that

$$f(na) = nf(a) + i.$$

Indeed $f(na) = nf(a) + i$ if and only if $\frac{i}{n} < [\alpha a] < \frac{i+1}{n}$.

(2) $[\alpha a] + [\alpha b] < 1$ if and only if $f(a+b) = f(b) + f(b)$.

(3) $[\alpha a] < [\alpha b]$ is equivalent to $f(b-a) = f(b) - f(a)$.

Proof. Easy to verify. □

The next lemma and its following corollary allow us to work also with rational multiples in $\mathcal{L}$ and hence to compare decimal parts in a more subtle way.

Lemma 2.2. *Let a, b and m be integers, and $n \in \mathbb{N}$.*

(1) There exists a quantifier-free formula $\varphi(x, y)$, depending on m and n, such that $\mathcal{Z}_\alpha \models \varphi(a, b)$ if and only if

$$[\alpha a] < \frac{m}{n} [\alpha b].$$

(2) Suppose that $\ell_1, \ell_2 \in \mathbb{Z}$, then there exists a quantifier-free formula $\varphi(x, y)$, depending on m, n, ℓ_1 and ℓ_2, such that $\mathcal{Z}_\alpha \models \varphi(a, b)$ if and only if

$$\ell_1 < n [\alpha a] + m [\alpha b] < \ell_2.$$

Proof. For part (1), first suppose that we have $0 < m < n$; other cases can be proved similarly or else they turn into triviality.

Note that there are natural numbers i and j with $0 \leq i < n$ and $0 \leq j < m$ such that $[\alpha a] \in (\frac{i}{n}, \frac{i+1}{n})$ and $[\alpha b] \in (\frac{j}{m}, \frac{j+1}{m})$. The latter conditions are $\mathcal{L}$-expressible using part (1) of Lemma 2.1. Also, observe that non-trivial cases only happen when $i < m$ and $j = i$; that is, both $[\alpha a]$ and $\frac{m}{n} [\alpha b]$ belong to the same interval $(\frac{i}{n}, \frac{i+1}{n})$. In this case, using parts (3) and (1) of Lemma 2.1, we can show that $n [\alpha a]$ is less than $< m [\alpha b]$ if and only if $f(mb - na) = mf(b) - nf(a)$.

For part (2), note that, depending on whether m is negative or positive, non-trivial cases occur only when we have that $-m \leq \ell_1 < \ell_2 \leq n$ or $0 \leq \ell_1 < \ell_2 \leq n + m$, respectively. Then, apply an argument similar to the proof of part (1). □

The following corollary is a consequence of part (2) of Lemma 2.2.

Corollary 2.3. *Let ℓ_1 and ℓ_2 be integers, and suppose that $\bar{a}, \bar{b}, \bar{m}$ and $\bar{n}$ are tuples of integers of length k. Then, there exists a quantifier-free $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, depending on $k, \ell_1, \ell_2, \bar{m}$ and $\bar{n}$, such that $\mathcal{Z}_\alpha \models \varphi(\bar{a}, \bar{b})$ if and only if*

$$\ell_1 < \sum_{i=1}^{k} n_i\,[\alpha a_i] + \sum_{i=1}^{k} m_i\,[\alpha b_i] < \ell_2. \tag{2.1}$$

Notation 2.4. From now on and by injecting suitable variables, we treat all the inequalities equivalent to those appeared in Lemmas 2.1 and 2.2 and their corollary as first-order $\mathcal{L}$-formulas. For example, by writing $\ell_1 < n\,[\alpha x]+m\,[\alpha y] < \ell_2$ we mean the quantifier-free $\mathcal{L}$-formula, $\varphi(x, y)$, obtained in part (2) of Lemma 2.2.

Using our observations so far and by applying our flexible notation, we are able to encode into our language some of the main features of the decimals involved in the structure $\mathcal{Z}_\alpha$.

Remark 2.5. Note again that to a decimal like $[\alpha a]$ nothing can be designated in $\mathcal{L}$. In fact, $\mathcal{L}$ is capable of describing these decimals merely in the cases that they participate in an inequality in the form of (2.1). However, as we will see, this amount of expressive power suffices for us to prove our results.

Based on the decimals that appear in our formulas, we consider two major cases in our argument towards proving model-completeness. In Section 3, we deal with formulas concerning the decimals of the form $[\alpha f^i(x)]$. We call these formulas *non-algebraic* (see Definition 3.4) whose properties appear in Axioms 1, 2 and 3 below.

Section 4 explores the properties of the *algebraic* formulas. Very briefly put, our algebraic formulas are very similar to the formula $f(x) = a$ which has finitely many solutions in any model of $\mathrm{Th}(\mathcal{Z}_\alpha)$ containing the parameter a (and indeed a unique solution when $\alpha > 1$). This is in contrast with a formula like $[\alpha x] \in (0, 1 - [\alpha a])$ which, according to Axiom 2 below, is satisfied by infinitely many elements x.

3. Non-algebraic formulas

As mentioned earlier, some of the essential properties of the function f will be described as a consequence of Kronecker's approximation lemma, and our aim is to exploit the full extent of this fact in our axiomatization. A proof of this theorem can be found in [HW08, Theorem 442].

Fact 3.1 (Kronecker's Approximation Lemma)**.**

Let $n \in \mathbb{N}$ be fixed and the real numbers $\beta_1, \ldots, \beta_n, 1$ be linearly independent over $\mathbb{Q}$. Then the set $\big\{([\beta_1 x], \ldots, [\beta_n x]) : x \in \mathbb{N}\big\}$ is dense in $(0, 1)^n$.

As an immediate consequence of Kronecker's lemma, it can be easily checked that $\mathcal{Z}_\alpha$ is a model of part (2) of the following axiom, and the first part can be verified similar to part (2) of Lemma 2.1.

Axiom 1.

(1) For all $x_1, \ldots, x_n$, there exists a unique j in $\{0, \ldots, n-1\}$ such that

$$j < \sum_{i=1}^{n} [\alpha x_i] < j + 1.$$

(2) The relation $[\alpha x] < [\alpha y]$ is a dense linear order.

Remark 3.2. An interesting, but not directly relevant, observation is that each $n\mathbb{Z}$ is dense in $\mathbb{Z}$ with respect to the order defined by the decimal parts. More generally, given any congruence class $n\mathbb{Z} + i$ and any interval $I = ([\alpha a], [\alpha b]) \subset (0, 1)$ there are infinitely many integers like c in that congruence class whose decimal part $[\alpha c]$ belongs to I. This can be proved using the ideas appeared in the proof of Proposition 1.2.

For given integers a and b, decimals other than $[\alpha a]$ and $[\alpha b]$ also intervene in investigating formulas like $f^n(a + b) = f^n(a) + f^n(b) + \ell$ where the exact value of ℓ is determined by the values of the decimals below:

$$[\alpha a], [\alpha b], [\alpha f(a)], [\alpha f(b)], \cdots, \left[\alpha f^{n-1}(a)\right], \left[\alpha f^{n-1}(b)\right].$$

To engage with decimals of the form $[\alpha f^i(a)]$, we aim to expand on the idea of the proof of Proposition 1.2 further as follows: Notice that we let $f^0(x) = x$. Since α is transcendental, each finite sequence $1, \alpha, \ldots, \alpha^n$ is $\mathbb{Q}$-linearly independent and this fact provides us, through the following extended version of Fact 3.1, with a more amount of "control" over the decimals of the form $[\alpha f^i(x)]$. The following theorem concerns the natural numbers and the word "dense" there has its usual meaning in the reals.

Theorem 3.3 (Extended Kronecker's Lemma)**.**
For every $n \in \mathbb{N}$, the following set of $(n+1)$-tuples is dense in $(0, 1)^{n+1}$:

$$\Big\{ \left([\alpha a], [\alpha f(a)], [\alpha f^2(a)], \cdots, [\alpha f^n(a)]\right) : a \in \mathbb{N} \Big\}.$$

Proof. Assume that $[\alpha a] \in \left(\frac{k}{\alpha^m}, \frac{k+1}{\alpha^m}\right)$ for some $m, k \in \mathbb{N}$ $(m \geq 2)$. Then

$$\frac{k}{\alpha^{m-1}} < \alpha [\alpha a] < \frac{k+1}{\alpha^{m-1}}.$$

Now if $[\alpha^2 a] > \frac{k+1}{\alpha^{m-1}}$, then

$$[\alpha f(a)] = \left[\alpha^2 a\right] - \alpha [\alpha a] \in (0, 1 - \frac{k+1}{\alpha^{m-1}}).$$

Similarly if $[\alpha^2 a] < \frac{k}{\alpha^{m-1}}$, then

$$[\alpha f(a)] = \left[\alpha^2 a\right] - \alpha\,[\alpha a] + 1 \in (1 - \frac{k}{\alpha^{m-1}}, 1).$$

Since $(0,1) \subseteq (0, 1 - \frac{k+1}{\alpha^{m-1}}) \cup (1 - \frac{k}{\alpha^{m-1}}, 1)$ by applying Kronecker's lemma to α and α^2 it is possible to find a such that $[\alpha a] \in (a,b)$ and $[\alpha f(a)] \in (c,d)$ for any a, b, c and d.

By induction and similar to the argument above, to control $[\alpha f^n(a)] = [\alpha f(f^{n-1}(a))]$ one needs to control $[\alpha^n f(a)]$. This is also possible, since $[\alpha^n f(a)] = [\alpha^{n+1} a - \alpha^n\,[\alpha a]]$. Now if $\alpha^n\,[\alpha a]$ is in an interval $(\frac{k}{\alpha^m}, \frac{k+1}{\alpha^m})$, one can control $[\alpha^{n+1} a]$ so that $[\alpha^n f(a)]$ belongs to a any desired interval. □

This extended Kronecker's lemma is included in our axioms as following.

Axiom scheme 2. For $n \in \mathbb{N}$ and any tuples of variables $\bar{y}$ and $\bar{z}$ with $|\bar{y}| = |\bar{z}| = n$ and $[\alpha z_i] < [\alpha y_i]$ for each $i \in \{1, \ldots, n\}$, there exist infinitely many x such that

$$\bigwedge_{i=1}^{n} \quad [\alpha y_i] < [\alpha f^i(x)] < [\alpha z_i].$$

Definition 3.4.

(1) Call a quantifier-free formula $\theta(x; \bar{y})$ *non-algebraic* if for some $\ell \in \mathbb{Z}$ and tuples of integers $\bar{n}$ and $\bar{m}$ it is equivalent to a formula of the form

$$\begin{aligned} n_0\,[\alpha x] + n_1\,[\alpha f(x)] + \ldots + n_k \left[\alpha f^k(x)\right] < \\ m_0\,[\alpha y_0] + \ldots + m_k\,[\alpha y_k] + \ell, \end{aligned} \tag{3.1}$$

where $|\bar{y}| = |\bar{n}| = |\bar{m}| = k+1$.

More generally define a *non-algebraic* formula $\theta(\bar{x}; \bar{y})$ with $|\bar{x}| = k$ and $|\bar{y}| = k'+1$ as a quantifier-free formula which is equivalent to a formula of the following form

$$\sum_{i=0}^{\ell_1} n_{1i} \left[\alpha f^i(x_1)\right] + \cdots + \sum_{i=0}^{\ell_k} n_{ki} \left[\alpha f^i(x_k)\right] < \sum_{i=0}^{k'} m_i\,[\alpha y_i] + \ell \tag{3.2}$$

for some $\ell_1, \ldots, \ell_k \in \mathbb{N}$ and $\ell, m_i, n_{ji} \in \mathbb{Z}$.

(2) Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and $\bar{a} \in M$. The *non-algebraic type* of $\bar{a}$ over A is a partial type consisting of all non-algebraic formulas $\theta(\bar{x}; \bar{b})$, with $|\bar{x}| = |\bar{a}|$ and $\bar{b} \in A$, which are satisfied by $\bar{a}$ in $\mathcal{M}$.

Remark 3.5.

(1) Non-algebraic formulas are closed under negation: Since we allow integers to appear in (3.2), namely the numbers ℓ, m_i and n_{ji}, the negation of a non-algebraic formula is a non-algebraic formula as well.

(2) Note that, to have the $\mathcal{L}$-formula expressing (3.2) above, we need to generalize Corollary 2.3 to obtain formulas with more than two variables; however, this can be done by a simple induction on the number of variables.

Lemmas 2.1 and 2.2 together with Corollary 2.3 can be used to have a better understanding about the non-algebraic type of a tuple over a set of parameters. In particular, a non-algebraic type $\pi(x;b)$ over a single parameter b determines, among other things, the value of the numbers $\ell_i \in \{0,1\}$ such that

$$\begin{aligned} f(x+b) &= f(x) + f(b) + \ell_1 \\ f^2(x+b) &= f(f(x)+f(b)+\ell_1) = f^2(x) + f(f(b)+\ell_1) + \ell_2 \\ f^3(x+b) &= f(f^2(x) + f(f(b)+\ell_1)+\ell_2) \\ &= f^3(x) + f(f(f(b)+\ell_1)+\ell_2) + \ell_3 \\ &\ \vdots \end{aligned} \tag{3.3}$$

By our notation, the set of equations above can be rewritten as the following set of inequalities assuming $\ell_i = 0$ (respectively $\ell_i = 1$).

$$\begin{aligned} &[\alpha x] + [\alpha b] < 1 && (> 1) \\ &[\alpha f(x)] + [\alpha(f(b)+\ell_1)] < 1 && (> 1) \\ &[\alpha f^2(x)] + [\alpha(f(f(b)+\ell_1)+\ell_2)] < 1 && (> 1) \\ &\vdots && \vdots \end{aligned}$$

Lemma 3.6 below provides a quantifier-free condition for a non-algebraic type to have a solution. To get the idea of its proof via an example, consider the following inequality in $\mathbb{R}$,

$$2z_0 + 3z_1 < m_0y_0 + m_1y_1 + 5,$$

where the appearing coefficients are of no specific importance. Observe that the existence of a solution $(z_0, z_1) \in (0,1)^2$ for this inequality is simply equivalent to the requirement that

$$m_0y_0 + m_1y_1 > -5.$$

Lemma 3.6. *Suppose that $\Gamma(x;\bar{y})$ is a finite set of non-algebraic formulas $\theta(x;\bar{y}')$ each in the form of (3.1) and with $\bar{y}' \subseteq \bar{y}$ and $|x| = 1$. Then, there exists a quantifier-free formula $\chi(\bar{y})$, depending on the numbers $k, \ell, \bar{n}$ and $\bar{m}$ appearing in the formulas $\theta \in \Gamma$, such that*

$$\mathcal{Z}_\alpha \models \forall \bar{y}\left(\exists x \Gamma(x;\bar{y}) \leftrightarrow \chi(\bar{y})\right).$$

Proof. In the case that $\Gamma(x;\bar{y})$ only consists of a single formula $\theta(x;\bar{y})$, let w denote the right hand-side of the inequality (3.1). Also, for each $i \in \{0,\dots,k\}$ let z_i denote $[\alpha f^i(x)]$. Now, fixing

w, note that the existence of a real-valued solution $(z_0, \cdots, z_k) \in (0,1)^{k+1}$ for the following linear inequality

$$\sum_{i=0}^{k} n_i z_i < w + \ell \tag{3.4}$$

reduces to whether the latter hyperplane intersects the regions $0 < z_i < 1$ for $i \in \{0, \ldots, k\}$ in $\mathbb{R}^{k+1}$. As in the example right before the lemma, the existence of such an intersection is equivalent to an inequality of the form $m'w < \ell'$, or more expanded:

$$m'_0 [\alpha y_0] + \ldots + m'_k [\alpha y_k] < \ell',$$

where the value of integers m', m'_i and ℓ' depend on the numbers n_i, m_i and ℓ appearing in (3.1). The latter inequality is the desired quantifier-free formula $\chi(\bar{y})$. We only need to check that the existence of a solution $(z_0, \ldots, z_k) \in (0,1)^{k+1}$ for (3.4) is equivalent to existence of an integer x with

$$\sum_{i=0}^{k} n_i \left[\alpha f^i(x)\right] < w + \ell,$$

and this can easily be verified using Theorem 3.3.

In the case that $\Gamma(x; \bar{y})$ contains more than one formula, we proceed as above by introducing the corresponding real variables z_i for the decimal part of different powers of $f(x)$, namely the decimals $[\alpha f^i(x)]$. This time, for each of the formulas $\theta(x; \bar{y}') \in \Gamma$ we need to consider a distinct real variable w_θ, like the variable w above, and the formula $\chi(\bar{y})$ must also contain a part for comparing these w_θ' s through a set of linear inequalities. □

Corollary 3.7. *Suppose that $\Gamma(\bar{x}; \bar{y})$ is a finite set of non-algebraic formulas $\theta(\bar{x}; \bar{y}')$ each in the form of (3.2) and with $\bar{y}' \subseteq \bar{y}$. Then, there exists a quantifier-free formula $\chi(\bar{y})$, depending on the numbers $\ell_1, \ldots, \ell_k, \ell, m_i, n_{ji}$ appearing in the formulas $\theta \in \Gamma$, such that the following holds in $\mathcal{Z}_\alpha$:*

$$\forall \bar{y} \left(\exists \bar{x} \Gamma(\bar{x}; \bar{y}) \leftrightarrow \chi(\bar{y}) \right). \tag{3.5}$$

Proof. Using Lemma 3.6 and an induction on the length of $\bar{x}$. □

Axiom scheme 3. All instances of formula (3.5) above when $\Gamma(\bar{x}; \bar{y})$ and $\chi(\bar{y})$ range over all formulas having the properties described in Corollary 3.7.

Notation 3.8.

(1) Let T_0 denote the theory of $\mathbb{Z}$-groups (that is Presburger arithmetic without order) in the language $\{0, 1, +, -\}$.
(2) Let $\mathcal{T}_{\text{nalg}}$, reads "$T$-non-algebraic", be T_0 together with Axioms 1 to 3.

Theorem 3.9. *Suppose that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are models of $\mathcal{T}_{\mathrm{nalg}}$ and let $A \subseteq M_1$ and $\bar{a} \in M_2$. Then, any finite fragment of the non-algebraic type of $\bar{a}$ over A is realized in $\mathcal{M}_1$.*

Proof. According to Axiom 3, the existence of a solution for each finite fragment of the non-algebraic type of a over A is equivalent to the satisfaction of a quantifier-free formula $\chi(\bar{y})$ by a finite tuple $\bar{b} \in A$. But, $\chi(\bar{b})$ holds in $\mathcal{M}_2$ if and only if it holds in $\mathcal{M}_1$. □

For an interesting connection of $\mathcal{T}_{\mathrm{nalg}}$ to o-minimality, see Subsection 5.3.

4. Adding algebraic formulas

In this section, we focus on the most general form of a finite set of $\mathcal{L}$-formulas by adding formulas of the form $H(x) = y$ where $H(x)$ is a term in the form of

$$\sum_{i=0}^{k} m_i f^i(x), \tag{4.1}$$

for some $m_i \in \mathbb{Z}$; we will refer to these integers as *coefficients of* $H(x)$.

Definition 4.1. An $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n; y)$ is called *algebraic* if it is equivalent to a formula of the following form

$$H_1(x_1) + \ldots + H_n(x_n) = y,$$

where $H_1(x_1), \ldots, H_n(x_n)$ are $\mathcal{L}$-terms each in the form of (4.1).

4.1. **One variable.**

For an algebraic formula $H(x) = c$ we add an axiom, among others, expressing the fact that there is a natural number K_H such that one in each K_H consecutive elements belongs to the range of $H(x)$. Assuming $\mathcal{M}_2 \models H(a) = c$ where c belongs to a submodel $\mathcal{M}_1$, the mentioned axiom helps find an approximate solution $a + j$ in $\mathcal{M}_1$ for some j less than K_H.

Lemma 4.2. *Let $H(x)$ be a term in the form of (4.1). Then, there exists a minimal natural number K_H, depending on coefficients appearing in (4.1), such that the following formulas hold* *in $\mathcal{Z}_\alpha$:*

$$\forall x \left(\bigvee_{j=0}^{K_H} H(-x) = -H(x) - j \right), \tag{4.2}$$

$$\forall x_1 x_2 \left(\bigvee_{j=0}^{K_H} H(x_1 + x_2) = H(x_1) + H(x_2) + j \right), \tag{4.3}$$

$$\forall x_1 x_2 \left(\bigvee_{j=0}^{K_H} H(x_2) = H(x_1) + j \ \rightarrow \ \bigvee_{j=0}^{K_H} |x_2 - x_1| = j \right), \tag{4.4}$$

$$\forall y \exists x \left(\bigvee_{j=0}^{K_H} H(x) = y + j \right). \tag{4.5}$$

Proof. Formulas (4.2), (4.3) and (4.4) can be proved by a double induction on the length of $H(x)$ and the greatest power of f which appears in it; the base case for (4.2) is obtained by the fact that $f(-x) = -f(x) - 1$ for any integer x, and for Formulas (4.3) and (4.4) one needs to use part (1) of Lemma 2.1.

Formula (4.5) can be proved using (4.3) and an induction on y. □

Axiom scheme 4. All instances of Formulas (4.2), (4.3), (4.4) and (4.5) for all $H(x)$ and K_H as described in Lemma 4.2.

Lemma 4.3. *Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be models of $\mathcal{T}_{\text{nalg}}$ together with Axiom scheme 4. Suppose that $\bar{b}, c \in M_1$, that $\Gamma(x; \bar{y})$ is a finite set of non-algebraic formulas, and that $H(x)$ is a term in the form of (4.1). If there is an element $a \in M_2$ that satisfies $\Gamma(x; \bar{b})$ and $H(x) = c$, then a belongs to M_1.*

Proof. What actually matters is that a satisfies $H(x) = c$ in $\mathcal{M}_2$. By (4.5) in Axiom scheme 4, there exist an integer $j \leq K_H$ and an element $a' \in M_1$ such that we have $H(a') = c + j$ in $\mathcal{M}_1$. By (4.4) in the same axiom, in $\mathcal{M}_2$ we have that $|a - a'| = j'$ for some $j' \leq K_H$. The axioms of T_0 ensure that a is the j'-th successor/predecessor of a', and this means that a is already a member of M_1. □

4.2. **Two variables.**

Lemmas 4.4 to 4.6 simply describe how one can take care of certain combinations of $\mathcal{L}$-terms by means of some non-algebraic formulas like (3.3). These lemmas are easy to verify but are given in full detail as they form a part of our axiomatization.

Lemma 4.4. *Suppose that $H(x)$ and K_H are as described in Lemma 4.2. Then, for each integer j with $|j| \leq K_H$ there is a finite set of parameter-free non-algebraic formulas $\Delta_H(x_1 x_2)$,* *depending on j and the coefficients appearing in $H(x)$, which determines the exact value of j in Formula (4.3). In other words, the following formula holds in $\mathcal{Z}_\alpha$:*

$$\forall x_1x_2\Big(\Delta_H(x_1x_2) \ \leftrightarrow\ H(x_1+x_2)=H(x_1)+H(x_2)+j\Big). \tag{4.6}$$

Axiom scheme 5. All instances of Formula (4.6) where $H(x), \Delta_H(x_1x_2)$ and j are as appeared in Lemma 4.4.

Lemma 4.5. *Suppose that $H_1(x)$ and $H_2(x)$ are two terms each in the form of (4.1). Then, there exist a term $H(x)$ and a natural number K_{H_1,H_2}, depending on coefficients appearing in $H_1(x)$ and $H_2(x)$, such that the following holds in $\mathcal{Z}_\alpha$:*

$$\forall x\left(\bigvee_{j,j'=0}^{K_{H_1,H_2}}\Big(H_1(H_2(x))=H(x)+j\ \wedge\ H_2(H_1(x))=H(x)+j'\Big)\right). \tag{4.7}$$

Moreover, for any given integers j, j' with $|j|, |j'| \leq K_{H_1,H_2}$ there exists a finite set of parameter-free non-algebraic formulas $\Delta_{H_1,H_2}(x)$, depending on j, j' and the coefficients appearing in $H_1(x)$ and $H_2(x)$, which determines the exact value of j and j' in (4.7).

Axiom scheme 6. All instances of Formula (4.7) where $H_1(x), H_2(x), H(x)$ and K_{H_1,H_2} are as described in Lemma 4.5.

Axiom scheme 7. All instances of the following formula where $H_1(x), H_2(x), H(x), K_{H_1,H_2}, \Delta_{H_1,H_2}(x), j$ and j' are as appeared in Lemma 4.5:

$$\forall x\left(\Delta_{H_1,H_2}(x)\ \rightarrow\ \big(H_1(H_2(x))=H(x)+j\wedge H_2(H_1(x))=H(x)+j'\big)\right).$$

Lemma 4.6. *Suppose that $H(x)$ is a term in the form of (4.1), and $\theta(x;\bar{y})$ is a non-algebraic formula in the form of (3.1). Then, there are finitely many non-algebraic formulas $\theta_1(x;\bar{y}),\dots,\theta_n(x;\bar{y})$, determined by the coefficients appearing in $H(x)$ and $\theta(x;\bar{y})$, such that the following holds in $\mathcal{Z}_\alpha$:*

$$\forall x\bar{y}\left(\theta\big(H(x);\bar{y}\big)\leftrightarrow\bigvee_{i=0}^{n}\theta_i(x;\bar{y})\right). \tag{4.8}$$

Axiom scheme 8. All instances of Formula (4.8) where $\theta(x;\bar{y}), \theta_1(x;\bar{y}),\dots,\theta_n(x;\bar{y})$ and $H(x)$ are as appeared in Lemma 4.6.

The following lemma forms our main technical step in solving systems involving more than one variable and consisting of at least one algebraic formula. It shows how a finite set of non-algebraic formulas constrained with a single algebraic formula can turn into a non-algebraic system of a smaller number of variables (which was already handled in Section 3). This is actually done in the expense of adding an extra parameter, the element a in the following lemma, which is obtained in the preimage of a new term $H(x)$ and from the parameters already available in the system.

For the sake of precision, the proof of the following lemma is detailed and involves a heavy use of notation. But the following general idea makes pursuing the proof easier: Assume that a finite set of non-algebraic formulas $\Gamma(x_1x_2;\bar{b})$ is given alongside with an algebraic equation $H_1(x_1)+H_2(x_2)=c$. We will first turn the latter into an equation $H(x_1+x_2)=c'$, and then to an equation $x_1+x_2=a$. Finally we replace x_2 in Γ with $a-x_1$, and we will finish with a totally non-algebraic system with a single variable.

Lemma 4.7 (Technical Trick)**.**

Let $\mathcal{M}$ be a model of $\mathcal{T}_{\text{nalg}}$ together with Axiom schemes 4-8. Suppose that $\bar{b},c\in M$, that $\Gamma(x_1x_2;\bar{b})$ is a finite set of non-algebraic formulas, and that $H_1(x_1)$ and $H_2(x_2)$ are two terms in the form of (4.1).

Then, there exist

(i) a term $H(x)$ in the form of (4.1),

(ii) an element $a\in M$ and an integer J with $H(a)=c+J$, and

(iii) finitely many finite sets of non-algebraic formulas $\Gamma_1(x;a\bar{b}),\dots,\Gamma_n(x;a\bar{b})$,

such that satisfiability of $\Gamma(x_1x_2;\bar{b})\cup\Big\{H_1(x_1)+H_2(x_2)=c\Big\}$ in $\mathcal{M}$ is equivalent to

$$\exists x\left(\bigvee_{i=0}^{n}\Gamma_i(x;a\bar{b})\right). \tag{4.9}$$

Proof. Suppose that $a_1,a_2\in M$ are some elements satisfying

$$\Gamma(x_1x_2;\bar{b})\cup\Big\{H_1(x_1)+H_2(x_2)=c\Big\}. \tag{4.10}$$

By (4.5) in Axiom 4, there are natural numbers J_1 and J_2, with $J_1\leq K_{H_1}$ and $J_2\leq K_{H_2}$, and elements $a_1',a_2'\in M$ such that we have

$$a_1=H_2(a_1')-J_2 \quad\text{and}\quad a_2=H_1(a_2')-J_1.$$

Hence, we have that

$$H_1(H_2(a_1')-J_2)+H_2(H_1(a_2')-J_1)=c. \tag{4.11}$$

Using Axiom 5 and by applying Lemma 4.4, there are finite sets of non-algebraic formulas $\Delta^1_{H_1}(x_1)$ and $\Delta^2_{H_2}(x_2)$ which determine the exact value of the integers j_1 and j_2 below

$$\begin{cases} H_1(H_2(a_1')-J_2)=H_1(H_2(a_1'))-H_1(J_2)+j_1,\\ H_2(H_1(a_2')-J_1)=H_2(H_1(a_2'))-H_2(J_1)+j_2.\end{cases}$$

Hence, (4.11) turns into

$$H_1(H_2(a_1'))-H_1(J_2)+j_1+H_2(H_1(a_2'))-H_2(J_1)+j_2=c. \tag{4.12}$$

Note that $\Delta^1_{H_1}(x_1)$ and $\Delta^2_{H_2}(x_2)$ are respectively a finite fragment of the non-algebraic type–over the empty set–of $H_2(a_1')$ and $H_1(a_2')$ in $\mathcal{M}$.

By applying Lemma 4.5 and by using Axioms 6 and 7, there are finite sets of non-algebraic formulas $\Delta^3_{H_1,H_2}(x_1)$ and $\Delta^4_{H_1,H_2}(x_2)$ which determine the exact value of the integers j_3 and j_4 in the following formulas

$$\begin{cases} H(a_1') = H_1(H_2(a_1')) + j_3, \\ H(a_2') = H_2(H_1(a_2')) + j_4, \end{cases}$$

where $H(x)$ denotes the term obtained in Lemma 4.5. This turns (4.12) into

$$H(a_1') + H(a_2') = c + H_1(J_2) - j_1 + j_3 + H_2(J_1) - j_2 + j_4. \tag{4.13}$$

Again, note that $\Delta^3_{H_1,H_2}(x_1)$ and $\Delta^4_{H_1,H_2}(x_2)$ are respectively a finite fragment of the non-algebraic type–over the empty set–of a_1' and a_2' in $\mathcal{M}$.

By applying Lemma 4.4 and Axiom 5 once again, there is a finite set of non-algebraic formulas $\Delta^5_H(x_1x_2)$ which determines the exact value of the integer j_5 in the formula below

$$H(a_1' + a_2') = H(a_1') + H(a_2') + j_5.$$

Hence, using (4.13), we have that

$$H(a_1' + a_2') = c + H_1(J_2) - j_1 + j_3 + H_2(J_1) - j_2 + j_4 + j_5.$$

Note that $\Delta^5_H(x_1x_2)$ is a finite fragment of the non-algebraic type–over the empty set–of the binary tuple (a_1', a_2') in $\mathcal{M}$. Let

$$\begin{cases} a := a_1' + a_2', \\ J := H_1(J_2) + H_2(J_1) - j_1 - j_2 + j_3 + j_4 + j_5. \end{cases}$$

Also, let x be a fresh variable. Then, the problem of satisfying (4.10) in $\mathcal{M}$ is equivalent to finding an element which satisfies

$$\begin{aligned} &\Gamma\Big(H_2(x) - J_2, H_1(a - x) - J_1; \bar{b}\Big) \cup \\ &\Delta^1_{H_1}\big(H_2(x)\big) \cup \Delta^2_{H_2}\big(H_1(a - x)\big) \cup \\ &\Delta^3_{H_1,H_2}(x) \cup \Delta^4_{H_1,H_2}(a - x) \cup \\ &\Delta^5_H(x, a - x). \end{aligned} \tag{4.14}$$

Since if $d \in M$ is a solution for (4.14), we can let the following elements serve as a solution for (4.10):

$$\begin{cases} a_1 := H_2(d) - J_2, \\ a_2 := H_1(a - d) - J_1. \end{cases}$$

By Lemma 4.6 and Axiom 8, there exist finite sets of non-algebraic formulas $\Gamma_1(x; a\bar{b}), \ldots, \Gamma_n(x; a\bar{b})$ such that the conjunction of the formulas in (4.14) is equivalent to

$$\bigvee_{i=0}^{n} \Gamma_i(x; a\bar{b}).$$

□

Lemma 4.8. *Suppose that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are models of $\mathcal{T}_{\mathrm{nalg}}$ together with Axiom schemes 4-8. Then, by the same assumption as in Lemma 4.7 with $\mathcal{M}$ replaced by $\mathcal{M}_1$, the following set of formulas is satisfiable in $\mathcal{M}_2$ if and only if it is satisfiable in $\mathcal{M}_1$:*

$$\Gamma(x_1 x_2; \bar{b}) \cup \Big\{ H_1(x_1) + H_2(x_2) = c \Big\}. \tag{4.15}$$

Proof. By applying Lemma 4.7 for $\mathcal{M}_2$, there are $a \in M_2$, a term $H(x) \in \mathcal{L}$, an integer J with $H(a) = c + J$, and finitely many sets of non-algebraic formulas $\Gamma_1(x; a\bar{b}), \ldots, \Gamma_n(x; a\bar{b})$ such that satisfiability of the set of formulas (4.15) in $\mathcal{M}_2$ is equivalent to a disjunction in the form of (4.9). Assuming (4.15) is satisfiable in $\mathcal{M}_2$, at least one of the mentioned sets, say $\Gamma_i(x; a\bar{b})$, is satisfiable in $\mathcal{M}_2$.

Since $c + J$ is an element of $\mathcal{M}_1$, we can use Lemma 4.3 to conclude that a is a member of $\mathcal{M}_1$ as well. Hence, by Theorem 3.9, the set of formulas (4.15) is also satisfiable in $\mathcal{M}_1$.

□

Remark 4.9. The content of Lemma 4.7 can be expressed by the first-order $\mathcal{L}$-formula below.

$$\forall \bar{y} z \exists w \bigvee_{J=-K}^{K} \Bigg(H(w) = z + J \ \wedge \ \Big(\exists x_1 x_2 \psi(x_1 x_2; \bar{y} z) \leftrightarrow \exists x \bigvee_{i=0}^{n} \Gamma_i(x; \bar{y} w) \Big) \Bigg),$$

where $K \in \mathbb{N}$ is determined by $\Gamma(x_1 x_2; \bar{y})$, $H_1(x_1)$ and $H_2(x_2)$; and $\psi(x_1 x_2; \bar{y} z)$ is the conjunction of formulas in (4.10) with $\bar{b}$ and c replaced by $\bar{y}$ and z respectively.

4.3. **More than two variables.**

The techniques and results of previous sections enable us to handle the case of more than two variables by carefully applying some suitable inductive proofs. For the sake of precision, the following lemmas are presented without leaving any detail unstated. However, the proofs are sketched as briefly as possible.

Using a simple induction we can prove the following generalization of Lemma 4.5.

Notation 4.10. Let $H_1(x), \ldots, H_n(x)$ be some terms each in the form of (4.1). For each $i \in \{1, \ldots, n\}$ the term $H_i^*(x)$ denotes the successive composition of all the terms in the set

$\{H_j(x) : j \neq i\}$. For example, $H_2^*(x)$ denotes

$$H_1(H_3(H_4(\cdots(H_n(x))\cdots))).$$

Lemma 4.11. *Suppose that $H_1(x), \ldots, H_n(x)$ are terms each in the form of (4.1). Then, there exist a term $H(x)$ and a natural number $K = K_{H_1,\ldots,H_n}$, depending on coefficients appearing in $H_1(x), \ldots, H_n(x)$, such that the following holds in $\mathcal{Z}_\alpha$:*

$$\forall x \left(\bigvee_{\bar{j} \in \mathbf{K}} \bigwedge_{i=1}^{n} \Big(H_i(H_i^*(x)) = H(x) + j_i \Big) \right), \tag{4.16}$$

where $\bar{j} = (j_1, \ldots, j_n)$ and $\mathbf{K} = \{0, \ldots, K\}^n$. Moreover, for any given integers $j_1, \ldots, j_n$ with $|j_1|, \ldots, |j_n| \leq K$ there exists a finite set of parameter-free non-algebraic formulas $\Delta(x) = \Delta_{H_1,\ldots,H_n}(x)$, depending on $j_1, \ldots, j_n$ and the coefficients appearing in $H_1(x), \ldots, H_n(x)$, such that $\mathcal{Z}_\alpha$ satisfies

$$\forall x \left(\Delta(x) \rightarrow \bigwedge_{i=1}^{n} \Big(H_i(H_i^*(x)) = H(x) + j_i \Big) \right). \tag{4.17}$$

Axiom scheme 9. All instances of Formula (4.16) where $H_1(x), \ldots, H_n(x), H(x)$ and K are as described in Lemma 4.11.

Axiom scheme 10. All instances of Formula (4.17) where $H_1(x), \ldots, H_n(x), H(x), K, \Delta(x), j_1, \ldots, j_n$ are as appeared in Lemma 4.11.

Lemma 4.12. *Let $n \in \mathbb{N}$ and $\mathcal{M}$ be a model of $\mathcal{T}_{\mathrm{nalg}}$ together with Axiom schemes 4-8. Suppose that $\bar{b}, c \in M$, that $\Gamma(\bar{x}; \bar{b})$ with $|\bar{x}| = n$ is a finite set of non-algebraic formulas, and that $H_1(x_1), \ldots, H_n(x_n)$ are some terms in the form of (4.1). Then, there exist*

(i) a term $H(x)$ in the form of (4.1),
(ii) an element $a \in M$ and an integer J with $H(a) = c + J$, and
(iii) finitely many finite sets of non-algebraic formulas $\Gamma_1(x_1 \cdots x_{n-1}; a\bar{b}), \ldots, \Gamma_m(x_1 \cdots x_{n-1}; a\bar{b})$,

such that satisfiability of $\Gamma(\bar{x}; \bar{b}) \cup \Big\{ H_1(x_1) + \cdots + H_n(x_n) = c \Big\}$ in $\mathcal{M}$ is equivalent to

$$\exists x_1 \cdots x_{n-1} \left(\bigvee_{i=0}^{m} \Gamma_i(x_1 \cdots x_{n-1}; a\bar{b}) \right). \tag{4.18}$$

Proof. The proof of this lemma is a generalization of the proof of Lemma 4.7. But we provide all the details to better clarify the argument.

Suppose that $a_1, \ldots, a_n \in M$ satisfy

$$\Gamma(x_1 \cdots x_n; \bar{b}) \cup \Big\{ H_1(x_1) + \cdots + H_n(x_n) = c \Big\}. \tag{4.19}$$

By (4.5) in Axiom 4, there are natural numbers $J_1^*, \ldots, J_n^*$, with $J_i^* \leq K_{H_i^*}$ for each i, and elements $a_1', \ldots, a_n' \in M$ such that for each $i \in \{1, \ldots, n\}$ we have

$$a_i = H_i^*(a_i') - J_i^*.$$

Hence, we have that

$$H_1(H_1^*(a_1') - J_1^*) + \cdots + H_n(H_n^*(a_n') - J_n^*) = c. \tag{4.20}$$

Using Axiom 5 and by applying Lemma 4.4, for each $i \in \{1, \ldots, n\}$ there is a finite set of non-algebraic formulas $\Delta_{H_i}^i(x_i)$ which determines the exact value of the integer j_i below

$$H_i(H_i^*(a_i') - J_i^*) = H_i(H_i^*(a_i')) - H_i(J_i^*) + j_i.$$

Hence, (4.20) turns into

$$\sum_{i=1}^{n} \Big(H_i(H_i^*(a_i')) - H_i(J_i^*) + j_i \Big) = c. \tag{4.21}$$

Note that each $\Delta_{H_i}^i(x_i)$ is a finite fragment of the non-algebraic type–over the empty set–of $H_i^*(a_i')$ in $\mathcal{M}$.

By applying Lemma 4.11 and by using Axioms 9 and 10, for each $i \in \{1, \ldots, n\}$ there is a finite set of non-algebraic formulas $\Delta_{H_1,\ldots,H_n}^i(x_i)$ which determines the exact value of the integer j_i^* in the following formula

$$H(a_i') = H_i(H_i^*(a_i')) + j_i^*,$$

where $H(x)$ denotes the term obtained in Lemma 4.11. This turns (4.21) into

$$\sum_{i=1}^{n} H(a_i') = c + \sum_{i=1}^{n} \Big(H_i(J_i^*) - j_i + j_i^* \Big). \tag{4.22}$$

Again, note that each $\Delta_{H_1,\ldots,H_n}^i(x_i)$ is a finite fragment of the non-algebraic type–over the empty set–of a_i' in $\mathcal{M}$.

By applying Lemma 4.4 and Axiom 5 once again, there is a finite set of non-algebraic formulas $\Delta_H(x_1 \cdots x_n)$ which determines the exact value of the integer j in the formula below

$$H(a_1' + \ldots + a_n') = \sum_{i=1}^{n} H(a_i') + j.$$

Hence, using (4.22), we have that

$$H(a_1' + \ldots + a_n') = c + \sum_{i=1}^{n} \Big(H_i(J_i^*) - j_i + j_i^* \Big) + j.$$

Note that $\Delta_H(x_1 \cdots x_n)$ is a finite fragment of the non-algebraic type–over the empty set–of the tuple $a'_1 \cdots a'_n$ in $\mathcal{M}$. Let

$$\begin{cases} a := a'_1 + \ldots + a'_n, \\ J := \sum_{i=1}^{n} \Big(H_i(J^*_i) - j_i + j^*_i \Big) + j. \end{cases}$$

Let $x_1, \ldots, x_{n-1}$ be some fresh variables and for each $i \in \{1, \ldots, n-1\}$ let x^*_i denote the term $H^*_i(x_i)$. Then, the problem of satisfying (4.19) in $\mathcal{M}$ is equivalent to finding $(n-1)$-many elements which satisfy

$$\begin{aligned} &\Gamma\Big(x^*_1 - J^*_1,\ \ldots,\ x^*_{n-1} - J^*_{n-1},\ H_n\big(a - \sum_{i=1}^{n-1} x_i\big) - J^*_n;\ \bar{b}\Big) \bigcup \\ &\bigcup_{i=1}^{n-1} \Delta^i_{H_i}(x^*_i) \bigcup \Delta^n_{H_n}\Big(H^*_n\big(a - \sum_{i=1}^{n-1} x_i\big)\Big) \bigcup \\ &\bigcup_{i=1}^{n-1} \Delta^i_{H_1,\ldots,H_n}(x_i) \bigcup \Delta^n_{H_1,\ldots,H_n}\big(a - \sum_{i=1}^{n-1} x_i\big) \bigcup \\ &\Delta_H\Big(x_1, \ldots, x_{n-1}, a - \sum_{i=1}^{n-1} x_i\Big). \end{aligned} \tag{4.23}$$

For if $d_1, \ldots, d_{n-1} \in M$ is a solution for (4.23), we can let the following elements serve as a solution for (4.19):

$$\begin{cases} a_1 & := H^*_1(d_1) - J^*_1, \\ \vdots & \\ a_{n-1} & := H^*_{n-1}(d_{n-1}) - J^*_{n-1}, \\ a_n & := H^*_n(a - \sum_{i=1}^{n-1} d_i) - J^*_n. \end{cases}$$

Lemma 4.6 can be easily generalized to non-algebraic formulas of more than one variables. Hence, similar to the proof of Lemma 4.7, we can find the sets of non-algebraic formulas desired by the lemma.

□

Lemma 4.13. *With the same assumptions as in Lemma 4.12 with $\mathcal{M}$ replaced by $\mathcal{M}_1$, the following set of formulas is satisfiable in $\mathcal{M}_2$ if and only if it is satisfiable in $\mathcal{M}_1$:*

$$\Gamma(x_1, \ldots, x_n; \bar{b}) \cup \Big\{ H_1(x_1) + \ldots + H_n(x_n) = c \Big\}. \tag{4.24}$$

Proof. Similar to the proof of Lemma 4.8. □

Notation 4.14. Let $\mathcal{T}_\alpha$ denote the theory consisting of $\mathcal{T}_{\text{nalg}}$ together with the axiom schemes 4 to 8.

Main Theorem 4.15. *$\mathcal{T}_\alpha$ is a complete, model-complete and decidable theory which axiomatizes the theory of $\mathcal{Z}_\alpha$. Moreover, $\mathcal{T}_\alpha$ has the strict order property.*

Proof. Model-completeness results from Lemma 4.13. Also, it is easy to verify that the axioms of $\mathcal{T}_\alpha$ are recursively enumerable. Hence, $\mathcal{T}_\alpha$ is a complete decidable theory. The strict order property results from the fact that the dense linear order $[\alpha x] < [\alpha y]$ is definable in $\mathcal{T}_\alpha$. □

Remark 4.16. Simpler forms of the techniques used in this paper are applied, among others, in [KZ22] to eliminate quantifiers for $\mathcal{T}_\alpha$ when α is the golden ratio. In this case, the specific formula that expresses being in the range of f, namely $\exists y f(y) = x$, is equivalent to the quantifier-free formula $x = f(f(x) - x + 1)$. This is mainly due to the fact that for any integer x we have $f^2(x) = f(x) + x - 1$, which in turn holds because of the algebraic dependence $\alpha^2 = \alpha + 1$. There does not seem to exist an easy way to eliminate quantifiers even for such a simple formula in the case of a transcendental α.

Remark 4.17. Our proofs in this paper can be checked to show an as-well *effective* model-completeness for the theory of $\mathcal{Z}_\alpha$. In fact, based on the proofs appeared in Section 3, one can use the effective quantifier elimination available in the ordered field of reals to effectively obtain an equivalent quantifier-free formula for any formula of the form $\exists\bar{x}\theta(\bar{x};\bar{y})$ where $\theta(\bar{x};\bar{y})$ is non-algebraic. Using formulas (4.3) and (4.4) in Axiom 4, one can effectively find a universal formula equivalent to $\exists x H(x) = y$ for $H(x)$ a term in the form of (4.1). For an example, when α equals Euler's number e the formula $\exists x(f(x) = y)$ is equivalent to

$$\forall x\Big(y - 1 \neq f(x) \wedge y + 1 \neq f(x)\Big).$$

For systems involving more than one existential variables and containing an algebraic formula, one can apply the proof of Lemma 4.7 to effectively reduce the number of existential variables.

5. Concluding remarks

5.1. **The case of an algebraic α.** The techniques used in this paper can be applied to obtain the same result for an algebraic α. In fact, when α satisfies an equation of a minimal degree like

$$\alpha^n + k_{n-1}\alpha^{n-1} + \ldots + k_0 = 0, \tag{5.1}$$

with integer coefficients, we can use (5.1) to calculate the value of a decimal $[\alpha f^m(x)]$, with $m \geq n$, in terms of the decimals

$$[\alpha f(x)], \ldots, \left[\alpha f^{n-1}(x)\right].$$

At the same time, each term $H(x)$ of the form (4.1) can be assumed to contain powers of f strictly less than n. Having made the mentioned adjustments, the rest of the argument can easily be carried out.

5.2. **On definable sets.** Based on the terminology used in [T08, Section 3.1], there appear three fundamental types of sets in various areas of mathematics: The "structured" sets, the "random" sets, and sets of a "hybrid" nature. Below, we make a concise clarification on this phenomena concerning the definable sets in $\mathcal{Z}_\alpha$.

If a power of f does not appear in an existential formula $\varphi(x)$ with a single free variable x, then the quantifier elimination available in Presburger arithmetic shows that $\varphi(x)$ is actually describing a congruence class to which x belongs. So in this case, $\varphi(x)$ defines an infinite arithmetic progression which is a typical example of a "structured" set by having a completely predictable behaviour.

On the contrary, Connell proved in [C60, Theorem 2] that no Beatty sequence with an irrational modulo can contain an infinite arithmetic progression. That is the set of solutions of a formula like $\exists y(x = f^n(y))$, that forms a typical example of an existential formula containing a power of f, cannot contain an infinite arithmetic progression. It is not clear to us if the same fact holds for formulas of the form $\exists y(x = f(y) + f^2(y))$ that contain an addition of terms; the latter question might be of interest from the perspective of additive combinatorics.

However in proposition below, and using Theorem 3.1, we prove that in the range of any term of one variable we can find finite arithmetic progressions of arbitrary large length. This slightly generalizes a similar result by Connell in [C60, Theorem 2].

Proposition 5.1. *Let $H(x) = \sum_{i=0}^{k} m_i f^i(x)$. For any natural number n there exists an arithmetic progression of length n in the range of H in $\mathcal{Z}_\alpha$.*

Proof. To have an arithmetic progression of length n, it suffices to find x and y such that for any $\ell \leq n$ we have that $H(x + \ell y) = H(x) + \ell H(y)$. And the latter holds whenever there are integers x and y such that $\mathcal{Z}_\alpha$ satisfies the following non-algebraic formula for any $1 \leq i \leq k$

$$f^i(x + ny) = f^i(x) + nf^i(y),$$

or equivalently whenever we have the following for any $0 \leq i \leq k - 1$

$$\mathcal{Z}_\alpha \models 0 < \left[\alpha f^i(x)\right] + n \left[\alpha f^i(y)\right] < 1.$$

But, Theorem 3.1 allows us to find x and y with the desired properties. ∎

A similar argument as in the proof of proposition above shows that for an existential formula $\varphi(x)$ of more than one existential variable, the set of solutions $\varphi(\mathcal{Z}_\alpha)$ contains arithmetic progressions of arbitrary finite lengths.

The latter observation shows that the formulas containing a power of f behaves more-or-less similar to prime numbers in that they do not contain infinite arithmetic progression whereas they do contain arbitrary long finite arithmetic progressions ([GT08]). However, Proposition 1.2 shows that such definable sets may differ from the primes in intersecting each congruence class at infinitely many points. But it seems reasonable to consider them as hybrid sets in $\mathcal{Z}_\alpha$ just like as we do for primes in $\mathbb{Z}$.

To sum up, the structured definable sets in $\mathcal{Z}_\alpha$ are disjoint-by-finite from the hybrid sets, and still another interesting phenomena occurs in $\mathcal{Z}_\alpha$ when we consider two mentioned types of sets from the perspective of the order topology available in $\mathcal{Z}_\alpha$ by the formula $[\alpha x] < [\alpha y]$. In fact, both the structured and hybrid sets find a uniform description in this topology by being simultaneously dense and co-dense there.

5.3. **A connection to o-minimality.** We show that the non-algebraic part of $\mathcal{T}_\alpha$, which we have denoted by $\mathcal{T}_{\text{nalg}}$ in Section 3, gives rise to an o-minimal theory that embodies its main features.

First for a model $\mathcal{M} \models \mathcal{T}_{\text{nalg}}$ we associate a structure $\mathcal{A}_\mathcal{M}$ in a language $\mathcal{L}^*$ that contains a set of predicates meant to capture the non-algebraic content of $\mathcal{M}$.

So let $\mathcal{L}^* = \left\{ <, P_{\bar{m},\bar{n},\ell} \right\}_{\bar{m},\bar{n},\ell\in\mathbb{Z}}$ where each $P_{\bar{m},\bar{n},\ell}$ accepts tuples of arity $|\bar{m}| + |\bar{n}|$. Fix some $\mathcal{M} \models \mathcal{T}_{\text{nalg}}$ and let $A_\mathcal{M}$ be the subset of (possibly non-standard) reals defined as

$$A_\mathcal{M} := \left\{ [\alpha a] \,\middle|\, a \in M \right\}.$$

For $[\alpha a_1], \ldots, [\alpha b_1], \ldots \in A_\mathcal{M}$, we let $P_{\bar{m},\bar{n},\ell}([\alpha a_1], \ldots, [\alpha b_1], \ldots)$ hold in $\mathcal{A}_\mathcal{M}$ if and only if

$$\mathcal{M} \models \sum m_i [\alpha a_i] < \sum n_i [\alpha b_i] + \ell. \tag{5.2}$$

Note in particular that $P_{\bar{1},\bar{1},0}([\alpha a], [\alpha b])$ holds in $\mathcal{A}_\mathcal{M}$ if and only if

$$\mathcal{M} \models [\alpha a] < [\alpha b].$$

That is, $P_{\bar{1},\bar{1},0}$ coincides with the relation $<$ in $\mathcal{A}_\mathcal{M}$. Hence by Axiom 1 this predicate defines a dense linear ordering on $\mathcal{A}_\mathcal{M}$.

Towards introducing T^*, we keep using the notation $[\alpha x]$ for elements of an arbitrary $\mathcal{L}^*$-structure $\mathcal{A}$. Also, for simplicity and particularly in axiom schemes (2) and (3) below, we keep thinking of predicates $P_{\bar{m},\bar{n},l}$ as if they are reflecting the content of the inequality appeared in (5.2), while we carefully have this reservation in mind that an expression like $\sum m_i [\alpha a_i]$ is, by itself, just meaningless in T^* and does not refer to an actual point.

Let T^* be the theory that describes the following:

(1) The relation $<$ is a dense linear order.
(2) The predicates $P_{\bar{m},\bar{n},\ell}$ are consistent with the usual addition and ordering of real numbers. That is T^* describes how elements can be moved from each side of (5.2) to the other. For example if $P_{\bar{2},\bar{1},0}(a,c)$ holds, then we have that $P_{\bar{1},\overline{1,-1},0}(a,c,a)$. This example reflects the content of the fact that $2[\alpha a] < [\alpha c]$ implies $[\alpha a] < [\alpha c] - [\alpha a]$ in real numbers.
(3) If $\sum m_i\,[\alpha a_i] < \sum n_i\,[\alpha b_i] + \ell < 1$ then there is $[\alpha x]$ such that

$$\sum m_i\,[\alpha a_i] < [\alpha x] < \sum n_i\,[\alpha b_i] + \ell.$$

Because of the density enforced on the predicates of $\mathcal{L}^*$ by the axioms (1) and (3) above, it is easy to verify the following proposition.

Proposition 5.2. *T^* admits quantifier elimination in $\mathcal{L}^*$.*

Now, for some model $\mathcal{M} \models \mathcal{T}_{\text{nalg}}$ it is easy to see that the associated $\mathcal{A}_{\mathcal{M}}$ is a model of T^*. On the other hand, T^* is *similar to* an o-minimal theory in the sense that any set defined by a formula $\varphi(x, \bar{a}\bar{b})$ is a finite union of *intervals* of the form below

$$\left\{x : \sum m_i\,[\alpha a_i] < m\,[\alpha x] < \sum n_i\,[\alpha b_i] + \ell\right\}.$$

But, as mentioned earlier, the endpoints of this *interval* are not some actual points in an arbitrary model of T^*. However, in each of the structures $\mathcal{A}_{\mathcal{M}}$ these endpoints turn out to be elements of the form $[\alpha a]$. Moreover, at the expense of adding/subtracting an integer value to/from ℓ, we can write $m\,[\alpha x]$ as $[m\alpha x]$ or equivalently as $[\alpha z]$ for some z in $\mathcal{M}$. That is, each $\mathcal{L}^*$-formula $\varphi(x, \bar{a}\bar{b})$ becomes equivalent to a finite disjunction of formulas of the form below in $\mathcal{A}_{\mathcal{M}}$:

$$[\alpha a] < [\alpha x] < [\alpha b].$$

Hence for any models $\mathcal{M}, \mathcal{N} \models \mathcal{T}_{\text{nalg}}$ the two associated structures $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{A}_{\mathcal{N}}$ are elementary equivalent since we are able to form a back-and-forth system between them. In other words, there exists a completion of T^* that is o-minimal and is determined by $\mathcal{T}_{\text{nalg}}$.

We finish by posing the following question which is seemingly a natural continuation of the results appeared in this paper:

Question. Is the structure $\langle \mathbb{Z}, <, +, -, 0, 1, f\rangle$, which is $\mathcal{Z}_\alpha$ augmented by the usual ordering of integers, decidable? Is it model-complete? Or, does it admit quantifier elimination in a naturally expanded language?

Acknowledgements. We would like to thank Philip Hieronymi, for generously offering several questions to work on (including this and a special case of it that the first and third author treated in their previous paper ([KZ22]). The second author was in part supported by a grant from IPM (No. 99030044). We are also thankful to the anonymous referee whose comments led to a much better version of this paper.

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